Modelling Delay Propagation in Railway Networks Using Closed Family of Distributions

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We present a new method to describe the long-run behaviour of a railway network in which moderate disturbances delay the traffic. The distribution functions of the source delays are supposed to be known and we calculate the distribution of the propagated delays across the network. In particular, we give conditions under which a stable operation of the network is possible.

Technically, we assume that all delays are stochastically independent, this allows to determine accumulation, propagation and reduction of delays by simple operations on the corresponding distribution functions. Certain (large) families of distributions are closed under these operations and allow an approximation of the results by Hyper-Erlang distributions. In a network that can be described by a *cycle-free* graph we may thus calculate the distributions of all propagated delays.

For networks *with cycles*, we give an iterative approximation algorithm for the long-run stationary distributions. Using ideas from queuing theory, we are able to show that this approximation leads to a stable situation if, in each cycle, the time buffers in the timetable are larger than the expected source delays and if delays propagated from one cycle to another are bounded.

The validity of this approach is shown by comparing the resulting distributions to values obtained by a Monte Carlo simulation for a realistic network.

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1 Introduction

Public transport systems such as railway systems are subject to many types of disturbances in their everyday operation. The operating companies have to find ways to compensate at least for the minor delays and to guarantee connections that run close to the schedule. In particular, they have to take care that small delays cannot accumulate and propagate through the network causing severe delays or even breakdowns.

In this paper we present a method that allows to determine the long-run probability distributions of propagated delays given the distribution of the initial delays. This result can e.g. be used to determine the size and allocation of time buffers in a time table that guarantee a stable operation or to evaluate the benefit of certain waiting time regulations.

The initial delays, also called *source delays*, are typically caused by some minor technical problems, by construction sites or by a congestion due to an unexpected number of passengers boarding a train. Such delays may accumulate along lines and they may be propagated to other trains e.g. if an undelayed connecting train has to wait for a delayed feeder train. These secondary delays are called *propagated delays*.

To prevent delays from spreading through the network, one could allocate additional time buffers in the timetable or decouple the lines, e.g. by simply not waiting for delayed feeders. Both these steps reduce delays in the network, but they increase the total travel time of passengers and the number of missed connections. Therefore, the construction of timetables and waiting time rules has to balance the reduction of delays on one side and the increase e.g. in travel times on the other side. Our results contribute to this task by allowing to calculate the longterm effects of buffers and waiting time rules on delays.

Note that in this context, we do not take into account big, singular delays, e.g. after some accident. For these, a singular recovery schedule has to be used, see e.g. Liebchen et al (2010), Schachtebeck and Schöbel (2010). Instead, we are concerned with the average performance of the network in the long run.

One way to study the behaviour of the network under delays is to turn it into a Monte Carlo simulation model. E.g., in Engelhardt-Funke and Kolonko (2004), simulations are used to find robust timetables in a heuristic optimization setup. A drawback of this approach is that one might have to run an extreme number of simulations until a reliable estimate of the long-run stationary behaviour of the network is obtained.

Another approach is to consider the network as a queuing system and to study its stationary behaviour, see e.g. Engelhardt-Funke and Kolonko (2002), Huisman and Boucherie (2001) or Huisman et al (2002). This approach seem to be applicable mostly to isolated parts of a network or under a random operation of trains.

In this paper we use a different approach that can be used for a more realistic periodic timetable: source delays are assumed to be random variables with given cumulative distribution functions (cdf) and the propagation is modelled by suitable operations on the cdfs, a similar approach has been proposed in Fuhr (2007), Kolonko (2007), Büker (2010), Büker and Wendler (2009), Meester and Muns (2007), and Berger et al (2011). Here, a main problem is to model the operations on the delay cdfs in an efficient way and to develop a practicable algorithm for the complete network.

This becomes much simpler if we assume that all delays of different trains that meet somewhere in the network are stochastically independent. Then the operations for accumulation, propagation and reduction on the level of cdfs are mainly convolution and multiplication. If we restrict the cdfs of the source delays to a family of distributions that is closed under these operations, we may derive the cdfs of propagated delays for all nodes of a cycle-free network. So-called theta-exponential polynomials may be used, see Fuhr (2007), Kolonko (2007) and Büker and Wendler (2009). In this paper, we assume the cdfs of the source delays to be Hyper-Erlang distributions which are only 'approximately closed' but offer advantages over more complex families. Note that in a cycle-free network all events may be ordered topologically, i.e. we are able to visit each node after we have visited all of its predecessors and have determined their delay distributions. This approach is presented in the first part of the paper.

Real networks, however, when modelled as event-activity-network (EAN), will contain many directed cycles of events (nodes) connected by activities (arcs), in particular by change activities between different lines. In this case no topological ordering of the events is possible. We suggest here a new iterative approach to approximate the long-run distribution of delays in cycles and in the connected component of the EAN containing the cycle. The approximation simulates the operation of the network starting with zero delays and then propagating the delays around the cycles. Following ideas from queuing theory, we are able to show that this approach leads to a stable operation of the network with limiting distributions of delays under the following two plausible stability conditions:

BALANCED CYCLES: within each cycle the sum of all buffers is larger than the sum of the expected delays, and

BOUNDED PROPAGATION: the delays propagated from outside into the cycle must be bounded.

The paper starts with a formal definition of the EAN we are using in Section 2, the definition of the stochastic model for the source delays and their propagation are given in Section 3. The calculation of the delay distributions using closed families of distributions is explained in Section 4 while the approximation of delays in cycles of the EAN and the convergence under suitable conditions are presented in Section 5. Section 6 gives some experimental results showing that the distributions we are deriving for a realistic network are in excellent accordance with the results from a Monte Carlo simulation. A final conclusion is given in Section 7. The technical proofs of the convergence in cycles are collected in the Appendix in Section 8.

2 The Network

We assume that we are given a description of the physical network consisting of stations, tracks and lines. It is operated under a periodic timetable π that schedules arrivals and departures of all lines every τ minutes. We replace this direct description by a more abstract model, the event-activity-network (EAN) that has become a standard in literature (see e.g. Nachtigall (1998), Liebchen and Möhring (2007)). It is a graph in which vertices are arrivals or departures of lines at or from a station. The directed arcs are drive-, stop- or change-activities connecting events. This is described more formerly in the following definitions.

Definition 1. The event-activity network $\mathfrak{N} = (\mathfrak{E}, \mathfrak{A})$ consists of

(i) the set of events $\mathfrak{E} = \mathfrak{E}^{arr} \cup \mathfrak{E}^{dep}$ with

- **arrival events** $arr(S,L) \in \mathfrak{E}^{arr}$ of a line L at a station S,
- **departure events** $dep(S,L) \in \mathfrak{E}^{dep}$ of a line L from a station S,

(ii) the set of activities $\mathfrak{A} = \mathfrak{A}^{stop} \cup \mathfrak{A}^{change} \cup \mathfrak{A}^{drive} \subset \mathfrak{E} \times \mathfrak{E}$ with

- $\mathfrak{A}^{stop}$ containing the **stop activities** $stop(S,L) := (arr(S,L), dep(S,L))$ that describe the halting of a train of line L in a station S,
- $\mathfrak{A}^{drive}$ containing the **drive activities** $drive(S,S',L) := (dep(S,L), arr(S',L))$ that describe the driving of a train of line L from station S to its next station S', and
- $\mathfrak{A}^{change}$ representing **change activities** $change(S,L,L') := (arr(S,L), dep(S,L'))$ that describe the possibility for passengers to change from a train of line L to one of line L' in station S.

While $\mathfrak{E}^{arr}, \mathfrak{E}^{dep}$ and $\mathfrak{A}^{stop}, \mathfrak{A}^{drive}$ represent the complete network, we allow $\mathfrak{A}^{change}$ to contain only a subset of possible change activities, namely those that should be maintained according to some waiting rules even if feeders are delayed. Typically, these are important connections used by many passengers. We assume that a reasonable selection of these change activities has been made. Note that in this context we do not use arcs representing constraints like minimal headways or order restrictions of trains.

Definition 2. Let an event-activity network $\mathfrak{N} = (\mathfrak{E}, \mathfrak{A})$ be given and let $\tau \in \mathbb{N}$ denote the fixed duration of a period.

a) A (τ-periodic) **timetable** is a vector $\pi = (\pi_e)_{e \in \mathfrak{E}}$ with $0 \le \pi_e < \tau$ for $e \in \mathfrak{E}$. π_e is the scheduled time for the event e to happen in each period.

b) Let $(l_a, u_a)_{a \in \mathfrak{A}}$ with $0 \le l_a \le u_a \le \infty$ be lower and upper bounds for the *duration* of activity $a \in \mathfrak{A}$. Then a timetable π is **feasible** if we have

$$[\pi_{e'} - \pi_e - l_a]_\tau \le u_a - l_a \quad \text{for all activities } a = (e,e') \in \mathfrak{A}. \tag{1}$$

Here, $[r]_\tau := \min\{r + z\tau \mid z \in \mathbb{Z}, r + z\tau \ge 0\}$ is the *periodic reduction modulo* τ for $r \in \mathbb{R}$.

c) For an activity $a = (e,e') \in \mathfrak{A}$

$$s(a) := [\pi_{e'} - \pi_e - l_a]_\tau \tag{2}$$

denotes the **time buffer** (also called *slack time*) for activity a contained in timetable π. Feasibility of the timetable therefore means that $s(a) \le u_a - l_a$.

According to the periodic timetable π, each event $e \in \mathfrak{E}$ takes place at times $z\tau + \pi_e, z = \dots, -1, 0, 1, \dots$. It is sometimes helpful to look at the time-expanded or 'rolled-out' network in which each such occurrence of e is an individual copy $e^{(z)} = (e, z)$ of event e. Then there is also an individual copy $a^{(z)} = (a, z)$ of activity a that connects a copy (e', z') of e' to $e^{(z)}$. Note that here the period z' of the corresponding starting event may be different from z, e.g. the drive activity $a^{(z)}$ may start with a departure in some period z' and arrive in a later period z. We agree to connect the individual activity to the scheduled period of its terminal event, hence $(a, z) = ((e', z'), (e, z))$.

3 Delays and Their Propagation

While the scheduled traffic is strictly periodic, source delays are typically caused by individual disturbances in each period. We now connect source delays to activities and determine the propagated delays observed at events.

Activity $a = (e', e) \in \mathfrak{A}$ has a minimal duration l_a and is subject to a random *source delay* $D_z(a)$ in the z-th period of its operation. The $(D_z(a))_{z \in \mathbb{Z}}$ are assumed to be independent, identically distributed (i.i.d) random variables with a cdf F_a on $[0, \infty)$. This means that we have $\mathbf{P}(D_z(a) \leq t) = F_a(t), t \geq 0$. Then the activity a in the z-th period actually needs the random time $l_a + D_z(a)$, i.e. the source delay describes the excess over the minimal duration. We assume that the delay distributions F_a for all $a \in \mathfrak{A}$ are given as input.

We then want to calculate the distribution of the secondary delays $X_z(e)$ that we see on events $e \in \mathfrak{E}$. $X_z(e)$ is the difference of the scheduled time $z\tau + \pi_e$ for that event in the z-th period and its actual time of occurrence. Note that we do not allow 'earliness', hence $X_z(e) \geq 0$. It will turn out that the $X_z(e), z \in \mathbb{Z}$, are either i.i.d. with a common cdf F_e or that they will converge with a limiting cdf F_e.

Next we define the mechanisms of delay propagation. Let $a_0 = (e_0, e) \in \mathfrak{A}$ be an activity with a source delay $D_z(a_0)$ during the z-th period and a buffer time $s(a_0)$. E.g. assume that a_0 is a stop activity that connects the arrival e_0 of some line L_0 at station S to its departure e and that this train has some technical problems during its stop in the z-th period causing an additional delay $D_z(a_0)$. The timetable schedules the departure e such the time between arrival and departure is $l(a_0) + s(a_0)$, making $s(a_0)$ a buffer time. Let $X_{z_0}(e_0)$ be the propagated delay at its arrival, see Figure 1.

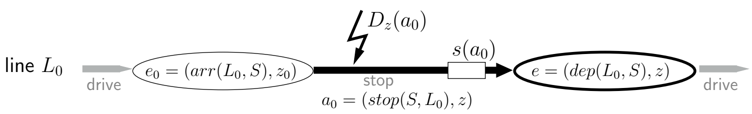

Figure 1: A delay $D_z(a_0)$ may add to the arrival delay $X_{z_0}(e_0)$ during activity a_0, $s(a_0)$ is the buffer time from the timetable

As we do not allow departure ahead of schedule, we have for the propagated delay of the

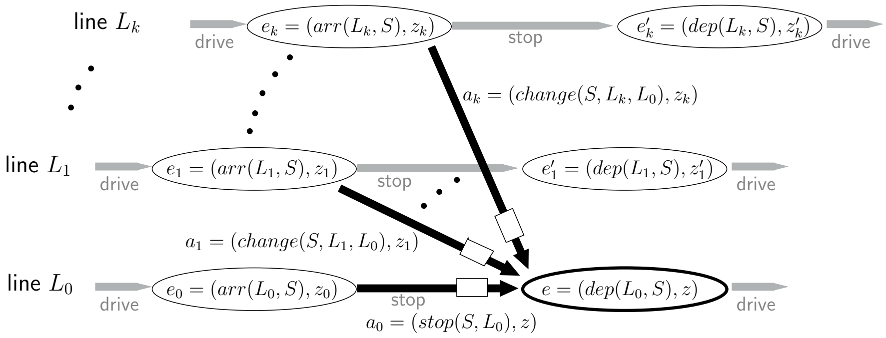

Figure 2: The departure event e has to wait for the end of all incoming activities $a_0, \ldots, a_k$. Here, the white rectangles indicate the buffer times $s(a_0), \ldots, s(a_k)$

departure

$$X_z(e) = propdel(a_0, z) := [X_{z_0}(e_0) + D_z(a_0) - s(a_0)]^+ \tag{3}$$

where $[x]^+ = \max\{x, 0\}$. The same formula may be applied to a drive or change activity to determine the delay at its terminal event given the delay at its starting event. Now assume that there are several activities $a_0, a_1, \ldots, a_k$ all terminating at event e, i.e. $a_l = (e_l, e), l = 0, \ldots, k$ for suitable events $e_0, \ldots, e_k$. E.g. assume that e is the departure of a train, a_0 is its halting activity and $a_1, \ldots, a_k$ are change activities from k feeder trains our train has to wait for as indicated in Figure 2. Given the individual propagated delays $X_{z_l}(e_l)$, $l = 0, \ldots, k$, of the starting nodes of all these activities, we see that the terminating event e in period z takes place after all these activities are finished, i.e.

$$\begin{aligned} X_z(e) &= \max\{propdel(a_l, z_l) \mid l = 0, 1, \ldots, k\} \\ &= \max\left\{[X_{z_l}(e_l) + D_z(a_l) - s(a_l)]^+ \mid l = 0, 1, \ldots, k\right\} \end{aligned} \tag{4}$$

Dropping the explicit reference to the period, we may say that if event e has to wait for k activities $a_i = (e_i, e), i = 0, \ldots, k$, then the actual delay of e will become the maximum of all $propdel(a_i)$, $i = 0, \ldots, k$.

In this case, a departure would wait for all feeder trains no matter how much they are delayed. In practice, there is a *waiting time rule* that states a train should not wait longer than κ time units for its (delayed) feeders, $0 \leq \kappa \leq \infty$. Here, the extreme cases $\kappa = 0$ and $\kappa = \infty$ are the 'no-wait' case and the 'all-wait' case. Again dropping the explicit period number, the propagated delay of e will then be

$$X(e) = \max\left\{propdel(a_0),\ \max\left\{propdel(a_i) \mid i = 1, \ldots, k, propdel(a_i) \leq \kappa\right\}\right\}. \tag{5}$$

Equation (5) is the general equation for delay propagation in EANs, given here for random variables. Formulation (5) assumes that the delays of the feeders are known to the connecting trains such that $propdel(a_i) \leq \kappa$ can be decided *before* the earliest time of departure $propdel(a_0)$.

The next step is to derive an algorithm that calculates the distribution of the $X(e)$ given the distribution of $D(a)$ and the $X(e_i)$ along Equation (5).

In (5), four operations are needed to determine $X(e)$: we have to add values, take their maximum, cut off negative values and find the max of values $\leq \kappa$. If we assume, that the delays interacting in (5) are independent, then these operations have simple counterparts on the level of distributions as is shown in Theorem 3 below. We therefore make the following assumption

Independent Delays For each event $e \in \mathfrak{E}$, all periods $z \in \mathbb{Z}$ and all activities $(a_l, z) = ((e_l, z_l), (e, z)), l = 1, \ldots, k$ terminating in (e, z) the following holds:
the delays $X_{z_1}(e_1), \ldots, X_{z_k}(e_k), D_z(a_1), \ldots, D_z(a_k)$ are independent.

If we restrict $D_z(a)$ to the typical small source delay originating from local disturbances, then it is reasonable that they are independent. Independence of propagated delays, however, may be violated. E.g., if a delayed feeder train A propagates its delay to trains B_1 and B_2 at a station S and if B_1, B_2 have a common station S' later on, then their delays there will not be independent as they both may carry delay from the common source A. However, in real networks such change configurations will be maintained probably only if stations S and S' are far apart. In this case, at least under the stability conditions we impose later, the common part of the delay from A will have vanished for B_1 and B_2 with a high probability. Meester and Muns (2007) discussed this issue and Büker and Seybold (2012) conclude from their experiments, that the error made by assuming independence is negligible. This is supported by our own results we show in Section 6 below.

The next Theorem gives the appropriate operations to calculate propagated delays under INDEPENDENT DELAYS.

Theorem 3. Let $X_1, \ldots, X_k$ be independent random variables with cdfs $F_1, \ldots, F_k$.

a) $Y := X_1 + X_2$ has cdf $\quad F_Y(t) := F_1 * F_2(t) := \int_0^\infty F_1(t-x) F_2(dx), \quad t \geq 0.$

b) $Y := \max\{X_1, \ldots, X_k\}$ has cdf $\quad F_Y(t) := \prod_{l=1}^k F_l(t), \quad t \geq 0.$

c) $Y := [X_1 - s]^+$ for some $s \geq 0$, has cdf $\quad F_Y(t) := F_1(t+s), \quad t \geq 0.$

d) $Y := \max\{X_l \mid X_l \leq \kappa, l = 1, \ldots, k\}$ for some $0 \leq \kappa \leq \infty$ has cdf
$F_Y(t) := \prod_{l=1}^k \big(1 - F_l(\kappa) + F_l(\min\{t, \kappa\})\big), \quad t \geq 0.$

Proof: a) - c) are standard results. For d), we have for $t \geq 0$

$$\begin{aligned}
\mathbf{P}\big(\max\{X_l \mid X_l \leq \kappa, l = 1, \ldots, k\} \leq t\big) &= \mathbf{P}\big(\max\{\mathbb{1}_{[X_l \leq \kappa]} \cdot X_l \mid l = 1, \ldots, k\} \leq t\big) \\
&= \prod_{l=1}^k \mathbf{P}\big(\mathbb{1}_{[X_l \leq \kappa]} \cdot X_l \leq t\big) = \prod_{l=1}^k \big(\mathbf{P}(X_l \leq \kappa, X_l \leq t) + \mathbf{P}(X_l > \kappa)\big) \\
&= \prod_{l=1}^k \big(F_l(\min\{t, \kappa\}) + 1 - F_l(\kappa)\big)
\end{aligned}$$

To be able to calculate the cdfs of propagated delays, we therefore have to find distributions on which we can perform the operations from Theorem 3 efficiently.

4 Closed Families of Distributions

We assume that all source delay distributions come from a single parametric family of distributions that has to fulfill certain conditions. First, it must be possible to fit distributions from that family to empirical source delay data. Secondly, there must be a simple way to obtain the results of operations as in Theorem 3 with adequate accuracy within that family.

In Thümmler et al (2005), an efficient way is given to fit a *Hyper-Erlang distribution* with a limited number of branches and stages to given data using a program called G-FIT. Different to that case, our empirical data may have a positive weight on 0, i.e. with a positive probability trains may be on time. This can be included into the (continuous) Hyper-Erlang distribution by adding just another branch to the distributions and adapting G-FIT accordingly (see Kirchhoff (2015) for details). We denote the family of Hyper-Erlang distributions extended in this way by *HypErl**.

Though *HypErl** fulfills the first condition, it is not closed under the operations required in Theorem 3. A class of distributions that has this closure property is the much larger family *ThetaExp* of cdfs F that can be written as *theta-exponential polynomials*, i.e. as

$$F(t) = \sum_{i=1}^{n} \mathbb{1}_{[\vartheta_i,\infty)}(t) a_i (t-\vartheta_i)^{k_i} e^{-\lambda_i(t-\vartheta_i)}, \quad t \geq 0,$$

for some $\vartheta_i \geq 0, a_i \in \mathbb{R}, \lambda_i \geq 0$ and $k_i \in \mathbb{N}_0$ for $i = 1,\ldots,n$ and $n \in \mathbb{N}$. The family *ThetaExp* was introduced in Trogemann and Gente (1997). We may embed *HypErl** into *ThetaExp*, so that *ThetaExp* seems to fulfill all requirements.

But, as was pointed out in Trogemann and Gente (1997), though operations like convolution and multiplication may be carried out within the family *ThetaExp*, resulting parameters get too complex and numerical problems may occur. We therefore use a hybrid technique to compute the convolution and products of delay distributions.

Essentially, we use the family *HypErl** for the delay distributions, starting with the source cdfs $F_a, a \in \mathfrak{A}$, which are assumed to be from *HypErl**. Each time a convolution or multiplication results in $F \in$ *ThetaExp* - *HypErl**, we approximate F by $\hat{F} \in$ *HypErl**. The approximation is based on the first three moments of F as described in Thümmler et al (2005), Johnson (1993) and Johnson and Taaffe (1989).

More precisely, if we have to calculate the distribution $F_0 := F_1 * F_2$ of the sum $X_1 + X_2$ with $F_1, F_2 \in$ *HypErl**, we take the first three moments of $X_1 + X_2$ and then determine the approximation $\hat{F}_0 \in$ *HypErl** according to these moments. Similarly, if we have to find $F_0 := F_1 \cdot F_2$ for $F_1, F_2 \in$ *HypErl**, we first use the *ThetaExp* family to find the precise solution $F_0 \in$ *ThetaExp* and then replace it by its approximation $\hat{F}_0 \in$ *HypErl**, again based on the first three moments which are also easy to determine for elements of *ThetaExp*. The approximation error can be estimated using methods from Meester and Muns (2007).

In this way, we are able to systematically determine the distribution for all events $e \in \mathfrak{E}$ if we can sort the events topologically, i.e. such that we are able to visit all predecessors of e in (5) before we visit e itself. This is the case if the following condition holds:

No-Cycle The EAN contains no directed cycles.

As we assumed the source delays $D_z(a), z \in \mathbb{Z}$, to be i.i.d., it then follows from an inductive argument that for all events $e \in \mathfrak{E}$ the propagated delays $X_z(e), z \in \mathbb{Z}$, will also be i.i.d with a common distribution function F_e that can be determined as described above.

However, different from condition **INDEPENDENT DELAYS**, condition **NO CYCLE** will not hold in most real networks as the change activities even if only maintained at a few important stations will lead to highly connected parts of the network.

5 Networks with Cycles

A subset of events $\mathfrak{E}' \subset \mathfrak{E}$ of the EAN is said to form a *strongly connected component* (component for short) if for each pair $e, e' \in \mathfrak{E}'$ there is a path of activities leading from e to e' and a path leading from e' to e. Then in particular, e, e' lie in a directed cycle. Within a strongly connected component of the network, we therefore cannot determine the delay distributions of all predecessors e_i of an event e before we visit e itself.

Any EAN can be split into a number of distinct strongly connected components $\mathfrak{E}_1, \ldots, \mathfrak{E}_m$. These components can be topologically ordered, i.e. we may assume that we visit component $\mathfrak{E}_l$ after we have visited all components $\mathfrak{E}_{l_1}, \ldots, \mathfrak{E}_{l_k}$ from which arcs (activities) lead into the present component $\mathfrak{E}_l$. In particular, we may assume that the distribution of all delays that may be propagated from the outside into $\mathfrak{E}_l$ have been determined before we start to examine $\mathfrak{E}_l$.

We fix an event $e \in \mathfrak{E}_l$ and want to determine the cdf F_e of its propagated delays. We assume that all delay distributions entering from outside the connected component have been determined already, so that we have i.i.d. delays on these arcs with a known distribution. These have been determined either by the method described in Section 3 or by the iterative method to be described now.

The iteration simulates the delay propagation within a cycle during the operation, starting with undelayed trains. In each iteration $n = 1, 2, \ldots$, an event e gets a delay $\Gamma_n(a_i)$ propagated from a predecessor event e_i in the EAN along activity $a_i = (e_i, e)$ as in (3), the only difference is that here for the predecessor e_i the delay $Y_{n-1}(e_i)$ is taken from the iteration before, as e may not have been visited yet in the present iteration. Therefore $propdel(a_i)$ is replaced by

$$\Gamma_n(a_i) := [Y_{n-1}(e_i) + D_n(a_i) - s(a_i)]^+ \tag{6}$$

here $D_n(a_i), n = 1, 2, \ldots$, are assumed to be i.i.d. with source delay cdf F_{a_i} and $s(a_i)$ is the time buffer on activity a_i as before. Now assume that the event e is the terminal event of activity $a_0 = (e_0, e)$ and that in addition, exactly the activities $a_i = (e_i, e), i = 1, \ldots, k$, may propagate delays to e, see Figure 2. Then we define similar to (5)

$$\begin{aligned} Y_n(e) &:= \max\left\{\Gamma_n(a_0),\ \min\{\kappa,\ \max\{\Gamma_n(a_i) \mid i=1,\dots,k\}\}\right\} \\ &= \max\Big\{[Y_{n-1}(e_0)+D_n(a_0)-s(a_0)]^+, \\ &\qquad \min\left\{\kappa, \max\{[Y_{n-1}(e_i)+D_n(a_i)-s(a_i)]^+ \mid i=1,\dots,k\}\right\}\Big\}, \\ &\text{for } n\geq 1 \text{ and } \quad Y_0(e') := 0 \text{ for all } e' \in \mathfrak{E}. \end{aligned} \tag{7}$$

Here $\kappa \in [0,\infty]$ is the time limit from the waiting time rule applied. In contrast to (5) we use a slightly different interpretation of κ: a train waits for its delayed feeder up to κ time units after its scheduled departure time. In (5), it was assumed that the train knows the delays of its feeders in advance and only waits for those that will arrive within the κ time limit. In a technical argument below, we have to assume the simpler waiting time rule contained in (7) as it preserves monotonicity.

As the operations in (7) are the same as in (5) we may use the operations of Theorem 3 to determine a cdf $F_{e,n}(t) := \mathbf{P}(Y_n(e)\leq t)$ from the cdfs $F_{e_i,n-1}$ of its predecessors from the last iteration, starting with $F_{e_i,0}(t)\equiv 1, t\geq 0$, for all $i=0,1,\dots,k$.

We are now going to show that the cdfs $F_{e,n}(t)$ determined in this way have a limit for $n\to\infty$ for all $e\in\mathfrak{E}_l$.

Theorem 4. Assume that the strongly connected components of the EAN are visited in a topological order and that for all activities a the delays $(D_n(a))_{n\geq 1}$ are i.i.d.. If (7) holds for all events e of the present component $\mathfrak{E}_l$ for some $\kappa=\kappa(e)\in[0,\infty]$, then

$$\mathbf{P}(Y_n(e)\leq t)\leq \mathbf{P}(Y_{n-1}(e)\leq t) \qquad \text{for all } n\geq 1, e\in\mathfrak{E}_l. \tag{8}$$

The *Proof* of this Theorem is given in the Appendix.

Theorem 4 shows that $n\to F_{e,n}(t)$ is monotonically decreasing for all $t\geq 0$ and therefore $F_{e,\infty}(t) := \lim_{n\to\infty} F_{e,n}(t)$ must exist. It may happen though, that the limit is not a proper distribution but has $\lim_{t\to\infty} F_{e,\infty}(t) < 1$ which means that with a positive probability, delays $Y_n(e)$ may grow to infinity during the iterations. The following Theorem gives conditions under which $\limsup_{n\to\infty} Y_n(e) < \infty$ with probability one. Together with Theorem 4 this guarantees a stable operation: the delay distributions converge to a proper distribution and we may use $F_e := F_{e,\infty}$ as the long-run distribution of event e.

Theorem 5 separates a single cycle from the present strongly connected component that contains the event e and analyses the long-run behaviour of delays within this cycle. Let this cycle consist of k events denoted by $e_1,e_2,\dots,e_k$ and activities $a_1,\dots,a_k$ as shown in Figure 3.

As this cycle is part of the EAN, at each event e_i, there may be additional delays summed up as $C(e_i)$ propagated from the outside. These may be delays from outside of the present connected component or from the component but from outside the present cycle.

In the separate cycle $e_1,\dots,e_k$ as depicted in Figure 3, we may simplify (7) to

$$Y_n(e_i) = \max\{C_n(e_i), [Y_{n-1}(e_{i-1})+D_n(a_{i-1})-s(a_{i-1})]^+\}, \quad i=1,\dots,k \tag{9}$$

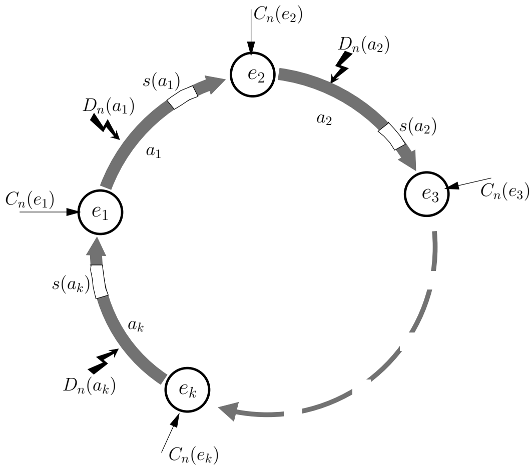

Figure 3: A cycle within a connected component with events $e_1, \ldots, e_k$ and k activities. Each activity a_i has a delay $D_n(a_i)$ and a time buffer $s(a_i)$. $C_n(e_1), \ldots, C_n(e_k)$ are delays propagated from outside the cycle

with starting values $Y_0(e_i) \equiv 0$ and $C_0(e_i) \equiv 0$ for $i = 1, \ldots, k$. Here, e_{i-1} and a_{i-1} have to be interpreted in a cyclic fashion, e.g. $e_{1-1} = e_k$.

The next Theorem states that these delays $Y_n(e_i)$ are bounded with probability one if the sum of the buffers in the cycle is larger than the sum of the expected source delays and if the propagated delays from outside the cycle are bounded.

Theorem 5. Assume as before that the delays $(D_n(a_i))_{n \geq 1}$ are i.i.d. for all activities in the cycle $e_1, \ldots, e_k$ and that the following two conditions hold

Balanced Cycle: $\displaystyle\sum_{i=1}^{k} \mathbf{E} D_n(a_i) < \sum_{i=1}^{k} s(a_i)$ and

Bounded Propagation: For each e_i there is a $\kappa(e_i) < \infty$ such that

$$C_n(e_i) \leq \kappa(e_i) \text{ for all } n \text{ with probability one.}$$

Then $\quad \mathbf{P}\left(\limsup_{n \to \infty} Y_n(e_i) < \infty\right) = 1 \quad$ for all $i = 1, \ldots, k$.

The quite technical *Proof* of this Theorem is given in the Appendix, it uses a similarity to queues as considered in Loynes (1962).

Note that we are considering here only one cycle of the present connected component. If we apply a waiting time rule with a finite κ (which may depend on the event) and if **BALANCED CYCLE** holds for *all* cycles of the component, then all delays in the component must stabilize during the iteration and their long term cdf can be calculated from our approximation.

The practical determination of the F_e within a connected component $\mathfrak{C}_l$ is as follows: in the n-th iteration we run through *all* events $e \in \mathfrak{C}_l$ updating $F_{e,n}$ from the $F_{e',n-1}, e' \in \mathfrak{C}_l$. We stop the iterations as soon as the difference $|F_{e,n} - F_{e,n-1}|$ falls below a given ε. Theorem 4 and 5 guarantee that this situation will occur and the practical experiments in the next Section show that the error we make seems negligible.

6 Experimental Results

We tested the results from our analytical approach against a Monte Carlo simulation. As an example network we used a simplified version of the Dutch Intercity network as described in Meester and Muns (2007), see Figure 4. We split the original four lines with two directions each into eight one-directional lines.

6.1 Network without Cycles

With the timetable and connections as given in Meester and Muns (2007), this network is cycle-free. In a first test set we therefore calculated the cdf F_e for each arrival event $e = arr(L,S)$ using the delay propagation (5) and the corresponding operations on cdfs as given in Theorem 3. As source delays on the tracks we chose Hyper-Erlang distributions with two branches and parameters adapted to those given in Meester and Muns (2007). Here we used $\kappa = \infty$ in (5), i.e. trains wait indefinitely long for their feeders.

For comparison we used an discrete event stochastic simulation based on the same EAN of the network that was used for the theoretical calculation. The source delays were drawn from the same Hyper-Erlang distributions that were used for the analytical model. In order to have reliable values for the comparison, we continued the simulation as long as the width of the 95%-confidence interval of the mean was more than 1% of its estimated value, resulting in more than 40 000 observations for each station on each line.

We compared the theoretical cdfs from our analytical approach to the empirical cdfs from the simulated values. In Figure 5 are two examples, the upper one shows almost complete agreement of the two cdfs whereas the lower one is one of the worst we have seen and has some deviation for small values. It is the station Amersfoort on line 5 where the approximation of the large probability of punctuality ($F_e(0) \approx 0.38$) was difficult for the theoretical approach.

In Figure 6 the empirical and theoretical quantiles of the delays are compared for all arrivals along two lines, the (abbreviated) stations are given on the x-axis. The boxplots show the simulated values, the lines give the lower and upper 25% quantile (dashed) and the median (dotted) from the analytically determined cdfs. Again, there is an excellent agreement between the two approaches.

6.2 Network with cycles

As the example network does not contain any cycles in the EAN, we added some more connections to be kept in stations $2,5,8$ and 7. The EAN now has six directed cycles consisting of the stations $2 \to 4 \to 5 \to 2$, $4 \to 7 \to 8 \to 5 \to 4$, and $2 \to 4 \to 7 \to 8 \to 5 \to 2$, each in both directions, the first two are indicated in Figure 4.

In each of these cycles we have to ensure the condition **BALANCED CYCLE**. In the timetable taken from Meester and Muns (2007) the cycle $2 \to 4 \to 5 \to 2$ e.g. contains time buffers that sum up to 18 minutes, our source delay distributions in that cycle have means that sum up to 11.50 such that $11.50 - 18 = -6.5 < 0$ as required. As we also have to guarantee **BOUNDED PROPAGATION**, we have to choose a finite waiting time limit, $\kappa := 20$ minutes in our case, which is large compared to the expected delays.

Figure 7 shows the simulation boxplots and the theoretical quantiles for the lines 3 and 5 as in Figure 6, this time with the artificial cycles. Again we have a remarkable agreement between the two approaches.

To demonstrate the impact of the cycles on the delays, Figure 8 shows a direct comparison of the average arrival delays with and without cycles. Shown is line 6 that runs right through the cycles and is therefore affected most. In the presence of cycles, average delays are higher in particular near the end of the line (station Utrecht) as was to be expected. Again an excellent agreement between the averages of the simulated delays and the expectations of the theoretical cdfs can be observed.

7 Conclusion

We have presented an analytical way to determine the cdfs of propagated delays in a railway network that may contain cycles. Within cycles we can show the convergence of an iterative approximation of the long-run equilibrium distribution provided the buffers in each cycle are larger than the expected delays and propagations from outside are limited. The comparison with simulated values shows a high accuracy of the results.

As most other authors we assume here that all delays 'meeting' at some station are independent. Also, we were not yet able to include delay propagation caused by minimal headways into our model. These restrictions will be the topic of our future research.

8 Appendix

8.1 Proof of Theorem 2

For non-negative valued random variables X, Y we say that X is stochastically smaller than Y ($X \preceq Y$) if

$$\mathbf{P}(Y \leq t) \leq \mathbf{P}(X \leq t), \quad t \geq 0.$$

We need the following Lemma, for a proof see e.g. Shaked and Shantikumar (1994), Theorem 1.A.3.

Lemma 6. Let U_i, V_i be real valued random variables with $U_i \preceq V_i,\ i = 1,2$. Assume that U_1, U_2 as well as V_1, V_2 are independent. Then the following holds:

$$\min\{U_1,U_2\} \preceq \min\{V_1,V_2\}, \max\{U_1,U_2\} \preceq \max\{V_1,V_2\}, \text{ and } U_1+U_2 \preceq V_1+V_2.$$

Note that this holds in particular if $U_1 \preceq V_1$ and U_2, V_2 are identically distributed or $U_2 = V_2 = \kappa$ for some $\kappa \in [0,\infty]$.

Theorem 4 claims that $n \mapsto \mathbf{P}(Y_n(e) \leq t)$ is decreasing for all $t \geq 0$ and all e in the present component.

Proof. of Theorem 4 We use induction on $n \geq 1$. For $n = 1$ we have $Y_0(e) \equiv 0$ for all e and hence $\mathbf{P}(Y_1(e) \leq t) \leq 1 = \mathbf{P}(Y_0(e) \leq t)$ for all $t \geq 0$ and for all events e of the present component.

Now assume that (8) holds for all $n' \leq n$ and we want to show it for $n+1$. Let $e_0, e_1, \ldots, e_k$ be the predecessor events of e from (7). Then e_i is either from the present component or from outside. If $a_i = (e_i, e)$ is an arc entering from outside, then the delays $(Y_l(e_i))_{l\geq 1}$ are i.i.d. with a distribution determined before the present component was entered, hence we have

$$\mathbf{P}(Y_n(e_i) \leq t) = \mathbf{P}(Y_{n-1}(e_i) \leq t), \quad t \geq 0 \quad \text{and} \quad Y_{n-1}(e_i) \preceq Y_n(e_i).$$

If, however, e_i lies *in* the present component, then $Y_{n-1}(e_i) \preceq Y_n(e_i)$ follows from the induction hypotheses. As the source delays $(D_n(a_i)_{n\geq 1}$ are assumed to be i.i.d., $\Lambda_n(a_i) := D_n(a_i) - s(a_i)$ are i.i.d. and therefore stochastically increasing and independent of $Y_{n-1}(e_i)$. Hence we may apply Lemma 6 to conclude that for $i = 0,1,\ldots,k$

$$\begin{aligned}
\mathbf{P}(\Gamma_{n+1}(a_i) \leq t) &= \mathbf{P}\big([Y_n(e_i) + D_{n+1}(a_i) - s(a_i)]^+ \leq t\big) \\
&= \mathbf{P}\big(\max\{0, Y_n(e_i) + D_{n+1}(a_i) - s(a_i)\} \leq t\big) = \mathbf{P}(Y_n(e_i) + D_{n+1}(a_i) - s(a_i) \leq t) \\
&\leq \ \mathbf{P}(Y_{n-1}(e_i) + D_n(a_i) - s(a_i) \leq t) = \mathbf{P}([Y_{n-1}(e_i) + D_n(a_i) - s(a_i)]^+ \leq t) \\
&= \mathbf{P}(\Gamma_n(a_i) \leq t).
\end{aligned}$$

From our general condition **Independent Delays** we see that $\Gamma_n(a_0), \ldots, \Gamma_n(a_k)$ are independent for all $n \geq 1$. By a repeated application of Lemma 6 we now obtain

$$\begin{aligned}
\mathbf{P}(Y_{n+1}(e) \leq t) &= \mathbf{P}\big(\max\big\{\Gamma_{n+1}(a_0), \min\{\kappa, \max\{\Gamma_{n+1}(a_i) \mid i = 1,\ldots,k\}\}\big\} \leq t\big) \\
&\leq \mathbf{P}\big(\max\big\{\Gamma_n(a_0), \min\{\kappa, \max\{\Gamma_n(a_i) \mid i = 1,\ldots,k\}\}\big\} \leq t\big) \\
&= \mathbf{P}(Y_n(e) \leq t).
\end{aligned}$$

Note that here we need the modified waiting rule to show with Lemma 6 that

$$\min\{\kappa, \max\{\Gamma_n(a_i) \mid i = 1,\ldots,k\}\} \preceq \min\{\kappa, \max\{\Gamma_{n+1}(a_i) \mid i = 1,\ldots,k\}\} \quad \square$$

$\square$

8.2 Proof of Theorem 5

As we assume that $C_n(e_i) \geq 0$ for all n and all $i = 1, \dots, k$, we may drop the $[\cdot]^+$ in equation (9) and simply write

$$Y_n(e_i) = \max\{C_n(e_i), Y_{n-1}(e_{i-1}) + D_n(a_{i-1}) - s(a_{i-1})\}, \quad i = 1, \dots, k. \tag{10}$$

The next Theorem resolves the recursion (9) resp. (10) into a form from which we can derive the bounds.

Theorem 7. Let (9) hold, then for all $1 \leq i \leq k$ and all $n \geq 1$

$$Y_n(e_i) = \max_{m=0,\dots,n} \left\{ C_{n-m}(e_{i-m}) + \sum_{l=1}^{m} \left(D_{n+1-l}(a_{i-l}) - s(a_{i-l}) \right) \right\} \tag{11}$$

where e_{i-m} and a_{i-l} are to be interpreted in a cyclic fashion, i.e. e_{i-m} is the event m steps backward from e_i in the cycle, see Figure 3.

Proof. Let $\Lambda_n(a_i) := D_n(a_i) - s(a_i)$. For $n = 1$ we have from (9) as $Y_0 \equiv 0$ and $C_n \geq 0$

$$Y_1(e_i) = \max\{C_1(e_i), Y_0(e_{i-1}) + \Lambda_1(a_{i-1})\} = \max\{C_1(e_i), \Lambda_1(a_{i-1})\}$$

which fulfills (11) for $n = 1$. Now assume (11) has been shown for n, then

$$\begin{aligned}
Y_{n+1}(e_i) &= \max\{C_{n+1}(e_i), Y_n(e_{i-1}) + \Lambda_{n+1}(a_{i-1})\} \\
&= \max\left\{C_{n+1}(e_i), \max_{m=0,\dots,n}\left\{C_{n-m}(e_{i-1-m}) + \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-1-l})\right\} + \Lambda_{n+1}(a_{i-1})\right\} \\
&= \max\Big\{C_{n+1}(e_{i-0}), \max_{m=0,\dots,n}\Big\{C_{n+1-(m+1)}(e_{i-(m+1)}) \\
&\qquad\qquad + \sum_{l=1}^{m} \Lambda_{n+2-(l+1)}(a_{i-(l+1)})\Big\} + \Lambda_{n+2-1}(a_{i-1})\Big\} \\
&= \max\Big\{C_{n+1}(e_{i-0}), \max_{m=1,\dots,n+1}\Big\{C_{n+1-m}(e_{i-m}) + \sum_{l=2}^{m} \Lambda_{n+2-l}(a_{i-l})\Big\} \\
&\qquad\qquad + \Lambda_{n+2-1}(a_{i-1})\Big\} \\
&= \max_{m=0,\dots,n+1}\left\{C_{n+1-m}(e_{i-m}) + \sum_{l=1}^{m} \Lambda_{n+2-l}(a_{i-l})\right\} \qquad \square
\end{aligned}$$

$\square$

With the abbreviation $\Lambda_n(i) = D_n(i) - s(i)$ we can now simplify (11) to

$$\begin{aligned}
Y_n(e_i) &= \max_{m=0,\dots,n}\left\{C_{n-m}(e_{i-m}) + \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-l})\right\} \qquad (12) \\
&= \max_{m=0,\dots,n}\left\{C_{n-m}(e_{i-m}) + \Lambda_{n-m+1}(a_{i-m}) + \Lambda_{n-m+2}(a_{i-(m-1)}) + \cdots + \Lambda_n(a_{i-1})\right\}
\end{aligned}$$

Theorem 5 states that $Y_n(e_i)$ is bounded in n for any e_i in the cycle.

Proof. *of Theorem 5*
1. From (12) we have

$$\limsup_{n\to\infty} Y_n(e_i) = \limsup_{n\to\infty} \max_{0\le m\le n} \Big\{ C_{n-m}(e_{i-m}) + \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-l}) \Big\}$$
$$\le \limsup_{n\to\infty} \max_{0\le m\le n} C_{n-m}(e_{i-m}) + \limsup_{n\to\infty} \max_{0\le m\le n} \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-l})$$

As the first term is bounded by assumption **BOUNDED PROPAGATION** we obtain

$$\mathbf{P}\Big(\limsup_{n\to\infty} Y_n(e_i) < \infty\Big) \ge \mathbf{P}\Big(\limsup_{n\to\infty} \max_{0\le m\le n} \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-l}) < \infty\Big)$$
$$= \mathbf{P}\Big(\limsup_{n\to\infty} \max_{1\le m\le n} \sum_{l=1}^{m} \Lambda_{l}(a_{i-l}) < \infty\Big), \tag{13}$$

where for the last equation we concluded from $(\Lambda_n(a_i))_{n\ge 1}$ i.i.d., that $\max_{0\le m\le n} \sum_{l=1}^{m} \Lambda_{n+1-l}(a_{i-l})$ and $\max_{1\le m\le n} \sum_{l=1}^{m} \Lambda_{l}(a_{i-l})$ have the same distribution. 2. Thus the Theorem is proven if we can show that

$$\mathbf{P}\Big(\limsup_{n\to\infty} \max_{1\le m\le n} \sum_{l=1}^{m} \Lambda_{l}(a_{i-l}) < \infty\Big) = 1 \tag{14}$$

for any $i = 1,\dots,k$. Note that for $m \ge k$ the sum $\sum_{l=1}^{m} \Lambda_l(a_{i-l})$ contains complete cycles of events $1,2,\dots,k$ plus a possibly incomplete cycle at the end. As the $\Lambda_n(a_i), n = 1,2,\dots$ on each activity are i.i.d., so are these cycles. Their long term behaviour is therefore given by the law of large numbers.

To make this more formal we write $(m)_k$ for the remainder of m/k such that $m = \lfloor \frac{m}{k} \rfloor k + (m)_k$. Let $K_\nu(i) := \sum_{l=1}^{k} \Lambda_{\nu\cdot k+l}(a_{i-l})$ denote the $\nu+1$-st complete cycle. Then we have

$$\sum_{l=1}^{m} \Lambda_l(a_{i-l}) = \sum_{\nu=0}^{\lfloor \frac{m}{k}\rfloor -1} \sum_{l=1}^{k} \Lambda_{\nu\cdot k+l}(a_{i-(\nu k+l)}) + \sum_{l=1}^{(m)_k} \Lambda_{\lfloor \frac{m}{k}\rfloor \cdot k+l}(a_{i-(\lfloor \frac{m}{k}\rfloor\cdot k+l)}))$$
$$= \sum_{\nu=0}^{\lfloor \frac{m}{k}\rfloor -1} \sum_{l=1}^{k} \Lambda_{\nu\cdot k+l}(a_{i-l}) + \sum_{l=1}^{(m)_k} \Lambda_{\lfloor \frac{m}{k}\rfloor \cdot k+l}(a_{i-l})$$
$$= \sum_{\nu=0}^{\lfloor \frac{m}{k}\rfloor -1} K_\nu(i) + \sum_{l=1}^{(m)_k} \Lambda_{\lfloor \frac{m}{k}\rfloor \cdot k+l}(a_{i-l}).$$

For any fixed $N \in \mathbb{N}$ we have

$$\begin{aligned}
\limsup_{n\to\infty} \max_{1\le m\le n} \sum_{l=1}^{m} \Lambda_l(a_{i-l}) &\le \limsup_{n\to\infty} \max_{1\le m\le n} \Big\{ \sum_{\nu=0}^{\lfloor \frac{m}{k} \rfloor -1} K_\nu(i) + \sum_{l=1}^{(m)_k} \Lambda_{\lfloor \frac{m}{k} \rfloor \cdot k+l}(a_{i-l}) \Big\} \\
&\le \max_{0\le L\le k-1} \limsup_{n\to\infty} \max_{1\le m\le n} \Big\{ \sum_{\nu=0}^{m-1} K_\nu(i) + \sum_{l=1}^{L} \Lambda_{mk+l}(a_{i-l}) \Big\} \\
&\le \max_{0\le L\le k-1} \sup_{m\ge 1} \Big\{ \sum_{\nu=0}^{m-1} K_\nu(i) + \sum_{l=1}^{L} \Lambda_{mk+l}(a_{i-l}) \Big\} \\
&\le \max_{0\le L\le k-1} \sup \Big\{ \sup_{1\le m<N} \big\{ \sum_{\nu=0}^{m-1} K_\nu(i) + \sum_{l=1}^{L} \Lambda_{mk+l}(a_{i-l}) \big\}, \qquad (15) \\
&\qquad\qquad \sup_{m\ge N} \big\{ \sum_{\nu=0}^{m-1} K_\nu(i) + \sum_{l=1}^{L} \Lambda_{mk+l}(a_{i-l}) \big\} \Big\}.
\end{aligned}$$

3. In (15), the first expression is finite, therefore the Theorem is proved, if we can show, that we may find $N \in \mathbb{N}$ such that for the second, infinite term in (15),

$$\sup_{m\ge N} \big\{ \sum_{\nu=0}^{m-1} K_\nu(i) + \sum_{l=1}^{L} \Lambda_{mk+l}(a_{i-l}) \big\} \Big\} < \infty \qquad (16)$$

with probability one.

4. To show this, we first apply the strong law of large numbers to the i.i.d. sequence $K_\nu(i)$. With probability one we have from assumption **Balanced Cycle**

$$\begin{aligned}
\lim_{m\to\infty} \frac{1}{m} \sum_{\nu=0}^{m-1} K_\nu(i) &= \mathbf{E}K_0(i) \\
&= \sum_{l=1}^{k} \big(\mathbf{E}D_l(a_{i-l}) - s(a_{i-l})\big) = \sum_{l=1}^{k} \big(\mathbf{E}D_1(a_l) - s(a_l)\big) < 0. \qquad (17)
\end{aligned}$$

Hence there is $\delta > 0$ such that with probability one the following holds: there is $N_1 \in \mathbb{N}$ such that for all $m \ge N_1$ we have

$$\frac{1}{m} \sum_{\nu=0}^{m-1} K_\nu(i) < -\delta < 0. \qquad (18)$$

As the sequence $\sum_{l=1}^{L} |\Lambda_{\nu k+l}(a_{i-l})|$ from the incomplete cycles is also i.i.d. for $\nu = 0, 1, \ldots,$ and integrable, we may again conclude with the strong law of large numbers that with probability one

$$\lim_{m\to\infty} \frac{1}{m} \sum_{\nu=1}^{m} \sum_{l=1}^{L} |\Lambda_{\nu k+l}(a_{i-l})| = \mathbf{E} \sum_{l=1}^{L} |\Lambda_l(a_{i-l})| < \infty$$

for any $L = 0, \dots, k-1$. But then it follows that we must have

$$\lim_{m\to\infty} \frac{1}{m} \sum_{l=1}^{L} |\Lambda_{mk+l}(a_{i-l})| = 0$$

and that with probability one there is $N_2 \in \mathbb{N}$ such that for $m \geq N_2$

$$\frac{1}{m} \sum_{l=1}^{L} |\Lambda_{mk+l}(a_{i-l})| < \delta. \tag{19}$$

But then we see from (18) and (19) that for $m \geq \max\{N_1, N_2\}$ we must have

$$\frac{1}{m} \sum_{\nu=0}^{m-1} K_\nu(i) + \frac{1}{m} \sum_{l=1}^{L} |\Lambda_{mk+l}(a_{i-l})| \ < \ -\delta + \delta = 0$$

from which (16) follows. □ □

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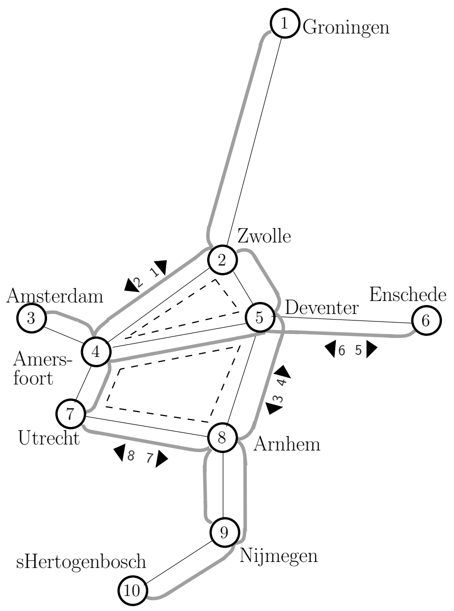


Figure 4: A simplified Dutch Intercity network, lines $1, \ldots, 8$ are shown in gray, the directions of the lines are indicated by black triangles. Two of the three artificial cycles used below are indicated by dashed lines

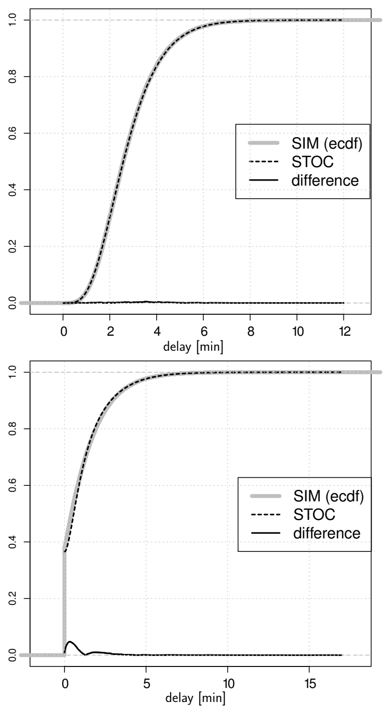


Figure 5: A comparison of the empirical cdf ”SIM (ecdf)” from simulated values and the theoretical cdf ”STOC” obtained by our approach. The upper is the arrival delay of line 4 at Arnhem, the lower the arrival delay of line 5 at Amersfort

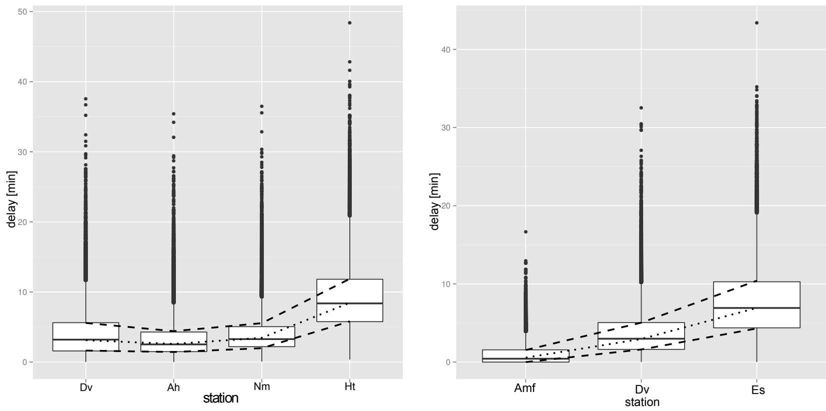


Figure 6: The boxplots summarize the simulated delays of line 3 (left) and line 5 (right). The dashed and dotted lines indicate the quantiles calculated from the theoretical cdfs F_e, dashed are lower and upper 25% quantile, the median is given by the dotted line

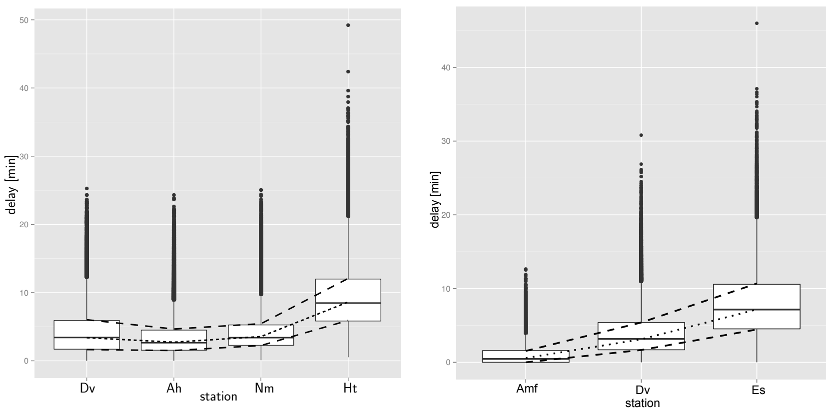
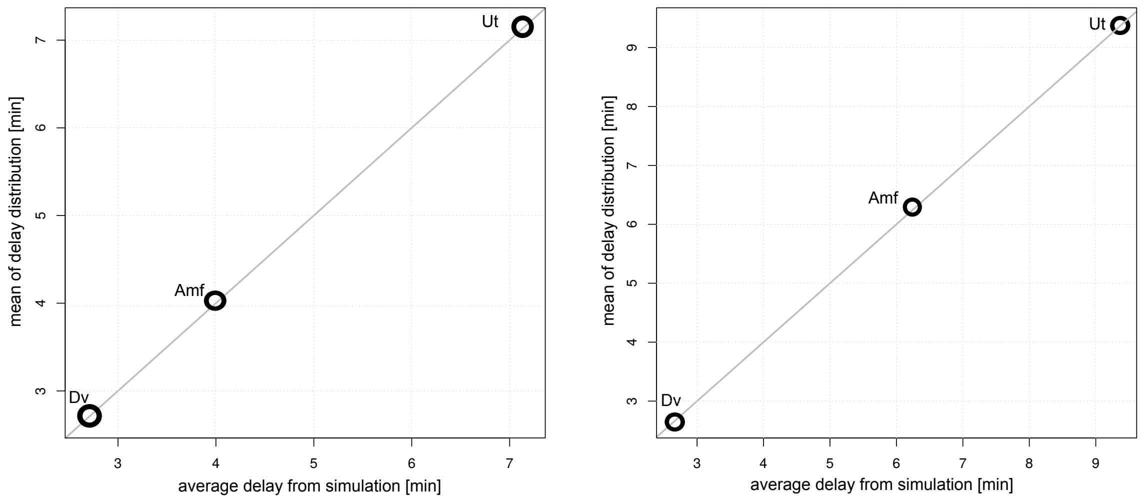


Figure 7: For lines 3 and 5 as in Figure 6, this plot shows the boxplots for the simulated delays and the quantiles of the delay distributions with additional connections that form cycles in the EAN



(a) without cycles

(b) with cycles

Figure 8: Scatterplots of simulated average delays and the corresponding means of the cdfs F_e of line 6